XVI. *Dr.* Halley's *Quadrature of the Circle improved: being a Transformation of his Series for that Purpose to others which converge by the Powers of* 80. *By the Rev.* John Hellins, *Vicar of* Potter's Pury, *in* Northamptonshire. *Communicated by* Nevil Maskelyne, *D. D. F. R. S. and Astronomer Royal.*

Read May 15, 1794.

1. DR. HALLEY'S method of computing the ratio of the diameter of the circle to its circumference was considered by himself, and other learned mathematicians, as the easiest the problem admits of. And although, in the course of a century, much easier methods have been discovered, still a celebrated mathematician of our own times has expressed an opinion, that no other aliquot part of the circumference of a circle can be so easily computed by means of its tangent as that which was chosen by Dr. HALLEY, viz. the arch of 30 degrees. This opinion, whether it be just or not, I shall not now inquire; my present design being to show, how the series by which Dr. HALLEY computed the ratio of the diameter to the circumference of the circle, may be transformed into others of swifter convergency, and which, on account of the successive powers of $\frac{1}{10}$ which occur in them, admit of an easy summation.

2. This transformation is obtained by means of different forms in which the fluents of some fluxions may be expressed.

To proceed with the greater clearness, I will here set down the fluxion in a general form, and its fluent, in the two series which are used in the following particular instance, and may be applied with advantage in similar cases.

3. The fluent of $\frac{x^{m-1}\dot{x}}{1-x^n}$ is $= \frac{x^m}{m} + \frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n} + \frac{x^{m+3n}}{m+3n}$, &c. which series, being of the simplest form which the fluent seems to admit, was first discovered, and probably is the most generally useful. But it has also been found, that the fluent of the same fluxion may be expressed in series of other forms, which, though less simple than that above written, yet have their particular advantages. Amongst those other forms of series which the fluent admits of, that which suits my present purpose is $\frac{x^m}{m.\overline{1-x^n}} - \frac{n x^{m+n}}{m.\overline{m+n}.\overline{1-x^n}|^2} + \frac{n.2n.x^{m+2n}}{m.\overline{m+n}.\overline{m+2n}.\overline{1-x^n}|^3} - \frac{n.2n.3n.x^{m+3n}}{m.\overline{m+n}.\overline{m+2n}.\overline{m+3n}.\overline{1-x^n}|^4}$ + &c. which, to say nothing of other methods, may easily be investigated by the rule given in page 64 of the third edition of EMERSON's Fluxions; or its equality with the former series may be proved by algebra.

4. On account of the sign — before x^n, in the last series, it may be proper to remark, that its convergency by a geometrical progression, will not cease till $\frac{x^n}{1-x^n}$ becomes $= 1$, or x becomes $= \sqrt[n]{\frac{1}{2}}$; and that, when x is a small quantity, and n a large number, this series will converge almost as swiftly as the former. For instance, if x be $= \sqrt{\frac{1}{3}}$, and $n = 8$, which are the values in the following case, the former series will converge by the quantity $x^n = \sqrt{\frac{1}{3}}|^8 = \frac{1}{81}$, and this series by the quantity $\frac{x^n}{1-x^n} = \frac{\frac{1}{81}}{1-\frac{1}{81}} = \frac{1}{80}$; where the difference in convergency will be but little, and the divisions by 80 easier than those by 81.

5. With respect to the indices m and n, as they are here supposed to be affirmative whole numbers, and will be so in the use I am about to make of them, the reader need not be detained with any observations on the cases in which these fluents will fail, when the indices have contrary signs.

6. It may be proper further to remark, that by putting $\frac{x^n}{1 - x^n} = z$, and calling the first, second, third, &c. terms of the series $\frac{x^m}{m.\overline{1 - x^n}} - \frac{n x^{m+n}}{m.\overline{m+n}.\overline{1 - x^n}|^2} + \frac{n.2n x^{m+2n}}{m.\overline{m+n}.\overline{m+2n}.\overline{1-x^n}|^3} + \&c.$ A, B, C, &c. respectively, the series will be expressed in the concise and elegant notation of Sir Isaac Newton; viz. $\frac{x^m}{m.\overline{1 - x^n}} - \frac{nzA}{m+n} + \frac{2nzB}{m+2n} - \frac{3nzC}{m+3n} + \&c.$ which is well adapted to arithmetical calculation.

7. I come now to the transformation proposed, which will appear very easy, as soon as the common series, expressing the length of an arch in terms of its tangent, is properly arranged.

If the radius of a circle be 1, and the tangent of an arch of it be called t, it is well known that the length of that arch will be $= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{t^{11}}{11} + \&c.$ Now, if the affirmative terms of this series be written in one line, and the negative ones in another, the arch will be

$$= \begin{cases} t + \frac{t^5}{5} + \frac{t^9}{9} + \frac{t^{13}}{13} + \frac{t^{17}}{17} + \&c. \\ - \frac{t^3}{3} - \frac{t^7}{7} - \frac{t^{11}}{11} - \frac{t^{15}}{15} - \frac{t^{19}}{19} - \&c. \end{cases}$$

And if, again, the first, third, fifth, &c. term of each of these series be written in one line, and the second, fourth, sixth, &c. in another, the same arch will be expressed thus:

$$= \left\{ \begin{array}{l} + \left\{ \begin{array}{l} t + \frac{t^9}{9} + \frac{t^{17}}{17} + \frac{t^{25}}{25} + \frac{t^{33}}{33} + \&c. \\ \frac{t^5}{5} + \frac{t^{13}}{13} + \frac{t^{21}}{21} + \frac{t^{29}}{29} + \frac{t^{37}}{37} + \&c. \end{array} \right. \\ \hline - \left\{ \begin{array}{l} \frac{t^3}{3} + \frac{t^{11}}{11} + \frac{t^{19}}{19} + \frac{t^{27}}{27} + \frac{t^{35}}{35} + \&c. \\ \frac{t^7}{7} + \frac{t^{15}}{15} + \frac{t^{23}}{23} + \frac{t^{31}}{31} + \frac{t^{39}}{39} + \&c. \end{array} \right. \end{array} \right.$$

All which series are evidently of the first form in article 3, and therefore their values may be expressed in the second form there given, or more neatly in the NEWTONIAN notation mentioned in art. 6. In each of these series the value of n is 8; and the value of m, $\left\{ \begin{array}{l} \text{in the first series, is } 1; \\ \text{in the second series, is } 5; \\ \text{in the third series, is } 3; \\ \text{in the fourth series, is } 7. \end{array} \right.$

If now we take $t = \sqrt{\frac{1}{3}}$, the tangent of 30°, which was chosen by Dr. HALLEY, we shall have the arch of 30°

$$= \left\{ \begin{array}{l} + \left\{ \begin{array}{l} \frac{1}{\sqrt{3}} \times : 1 + \frac{1}{9.81} + \frac{1}{17.81^2} + \frac{1}{25.81^3} + \frac{1}{33.81^4}, \&c. \\ \frac{1}{9\sqrt{3}} \times : \frac{1}{5} + \frac{1}{13.81} + \frac{1}{21.81^2} + \frac{1}{29.81^3} + \frac{1}{37.81^4}, \&c. \end{array} \right. \\ \hline - \left\{ \begin{array}{l} \frac{1}{3\sqrt{3}} \times : \frac{1}{3} + \frac{1}{11.81} + \frac{1}{19.81^2} + \frac{1}{27.81^3} + \frac{1}{35.81^4}, \&c. \\ \frac{1}{27\sqrt{3}} \times : \frac{1}{7} + \frac{1}{15.81} + \frac{1}{23.81^2} + \frac{1}{31.81^3} + \frac{1}{39.81^4}, \&c. \end{array} \right. \end{array} \right.$$

Six times this quantity will be = the semicircumference when radius is 1, and = the whole circumference when the diameter is 1. If therefore we multiply the last series by 6, and write $\sqrt{12}$ for $\frac{6}{\sqrt{3}}$, and express their value in the form given in art. 6, we shall have the circumference of a circle whose diameter is 1,

$$= \left\{ \begin{array}{l} + \left\{ \begin{array}{l} \frac{81\sqrt{12}}{80} - \frac{8A}{9.80} + \frac{16B}{17.80} - \frac{24C}{25.80} + \frac{32D}{33.80}, \&c. \\ \frac{81\sqrt{12}}{5.9.80} - \frac{8A}{13.80} + \frac{16B}{21.80} - \frac{24C}{29.80} + \frac{32D}{37.80}, \&c. \end{array} \right. \\ - \left\{ \begin{array}{l} \frac{81\sqrt{12}}{3.3.80} - \frac{8A}{11.80} + \frac{16B}{19.80} - \frac{24C}{27.80} + \frac{32D}{35.80}, \&c. \\ \frac{81\sqrt{12}}{7.27.80} - \frac{8A}{15.80} + \frac{16B}{23.80} - \frac{24C}{31.80} + \frac{32D}{39.80}, \&c. \end{array} \right. \end{array} \right.$$

8. All these new series, it is evident, converge somewhat swifter than by the powers of 80. For in the first series, which has the slowest convergency, the coefficients $\frac{8}{9}, \frac{16}{17}, \frac{24}{25}$, &c. are each of them less than 1; so that its convergency is somewhat swifter than by the powers of 80.

9. But another advantage of these new series is, that the numerator and denominator of every term except the first, in each of them, is divisible by 8; in consequence of which the arithmetical operation by them is much facilitated, the division by 80 being exchanged for a division by 10, which is no more than removing the decimal point. These series then, when the factors which are common to both numerators and denominators are expunged, will stand as below, (each of which still converging somewhat quicker than by the powers of 80), and we shall have the circumference of a circle whose diameter is 1,

$$= \left\{ \begin{array}{l} + \left\{ \begin{array}{l} \frac{81\sqrt{12}}{80} - \frac{A}{9.10} + \frac{2B}{17.10} - \frac{3C}{25.10} + \frac{4D}{33.10}, \&c. \\ \frac{9\sqrt{12}}{400} - \frac{A}{13.10} + \frac{2B}{21.10} - \frac{3C}{29.10} + \frac{4D}{37.10}, \&c. \end{array} \right. \\ - \left\{ \begin{array}{l} \frac{9\sqrt{12}}{80} - \frac{A}{11.10} + \frac{2B}{19.10} - \frac{3C}{27.10} + \frac{4D}{35.10}, \&c. \\ \frac{3\sqrt{12}}{7.80} - \frac{A}{15.10} + \frac{2B}{23.10} - \frac{3C}{31.10} + \frac{4D}{39.10}, \&c. \end{array} \right. \end{array} \right.$$

By which series the arithmetical computation will be much more easy than by the original series.

April 30, 1794. J. HELLINS.